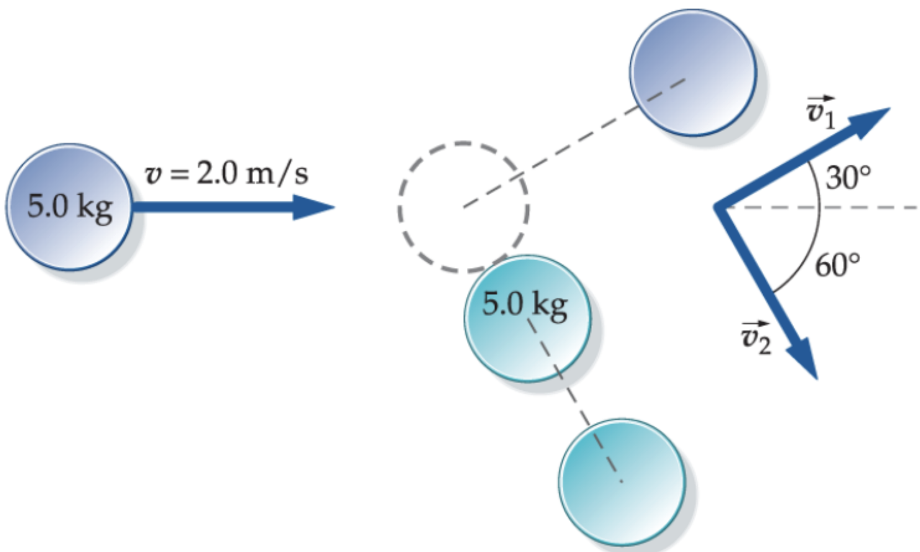
Exam 3 Solutions

Instructions

- Write your name on this test booklet.
- You are allowed a writing utensil, a calculator, and the PHY 201 formula sheet.
- You will have 110 minutes to complete this exam.
- Show all of your work.
- Good luck!

Problem 1

(40 points) A puck of mass 5.0 kg moving at $2.0\hat{x}$ m/s approaches an identical puck that is stationary on frictionless ice. After the collision, the first puck leaves with a speed v_1 at 30° to the original line of motion while the second puck leaves with speed v_2 at 60°, as shown in the diagram below.



- Calculate the speed of the first puck (v_1) after the collision.

See the next part below.

- Calculate the speed of the second puck (v_2) after the collision.

We can use the law of conservation of momentum to solve this problem. Recall that the total initial momentum of the system must be equal to the total final momentum of the system:

$$\vec{p}_i = \vec{p}_f$$

The final momenta of the two pucks are each going to have x and y components. The expression for momentum along the x-axis will be:

$$(5.0\,kg)(2.0\,m/s)\,\hat{x} + (5.0\,kg)(0.0\,m/s)\,\hat{x} = (5.0\,kg)(v_1)cos(30^\circ)\,\hat{x} + (5.0\,kg)(v_2)cos(60^\circ)\,\hat{x}$$

$$10\hat{x} = \frac{5\sqrt{3}v_1}{2}\,\hat{x} + \frac{5v_2}{2}\,\hat{x}$$

$$4 = \sqrt{3}v_1 + v_2$$

The expression for momentum along the y-axis will be:

$$(5.0\,kg)(0.0\,m/s)\,\hat{y} + (5.0\,kg)(0.0\,m/s)\,\hat{y} = (5.0\,kg)(v_1)sin(30°)\,\hat{y} - (5.0\,kg)(v_2)sin(60°)\,\hat{y}$$

$$0\,\hat{y} = \frac{5v_1}{2}\,\hat{y} - \frac{5\sqrt{3}v_2}{2}\,\hat{y}$$

$$0 = v_1 - \sqrt{3}v_2$$

$$v_1 = \sqrt{3}v_2$$

Plugging the second expression into the first gives:

$$4 = \sqrt{3}\left(\sqrt{3}v_2\right) + v_2$$

$$4 = 3v_2 + v_2$$

$$4 = 4v_2$$

$$v_2 = 1.0\,m/s$$

Then, we can solve for v_1:

$$v_1 = \sqrt{3}\,(1\,m/s) = \sqrt{3}\,m/s \approx 1.7\,m/s$$

- Calculate the velocity of the center of mass of the system before the collision.

The velocity of the center of mass of the system is given by:

$$\vec{v}_{COM} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i}$$

Plugging in values for the masses gives:

$$\vec{v}_{COM} = \frac{(5.0\,kg)(2\,m/s)\hat{x} + (5.0\,kg)(0.0\,m/s)\hat{x}}{5\,kg + 5\,kg}$$

$$\vec{v}_{COM} = 1.0\hat{x}\,m/s$$

- Calculate the velocity of the center of mass of the system after the collision.

We could go through the full calculation for the center of mass in a similar manner as the previous part. Let's start with the x-direction:

$$\vec{v}_{COM,x} = \frac{\sum_i m_i \vec{v}_{i,x}}{\sum_i m_i}$$

$$\vec{v}_{COM,x} = \frac{(5.0\ kg)\left(\sqrt{3}cos(30^\circ)\ m/s\right) + (5.0\ kg)\left(cos(60^\circ)\ m/s\right)}{5\ kg + 5\ kg}$$

$$\vec{v}_{COM,x} = \frac{(5.0\ kg)\left(\frac{3}{2}\ m/s\right) + (5.0\ kg)\left(\frac{1}{2}\ m/s\right)}{10\ kg}$$

$$\vec{v}_{COM,x} = \frac{10\ kg \cdot m/s}{10\ kg}$$

$$\vec{v}_{COM,x} = 1.0\,m/s$$

And the y-direction:

$$\vec{v}_{COM,y} = \frac{\sum_i m_i \vec{v}_{i,x}}{\sum_i m_i}$$

$$\vec{v}_{COM,y} = \frac{(5.0\ kg)\left(\sqrt{3}sin(30^\circ)\ m/s\right) - (5.0\ kg)\left(sin(60^\circ)\ m/s\right)}{5\ kg + 5\ kg}$$

$$\vec{v}_{COM,y} = \frac{(5.0\ kg)\left(\frac{\sqrt{3}}{2}\ m/s\right) - (5.0\ kg)\left(\frac{\sqrt{3}}{2}\ m/s\right)}{10\ kg}$$

$$\vec{v}_{COM,y} = 0\,m/s$$

But you really don't have to do any calculations here. From our discussions about conservation of momentum and collisions, we know that the velocity of the center of mass does not change before and after a collision. So you could have just answered "the same as the previous part".

- Calculate the total kinetic energy of the system before the collision.

Let's sum the kinetic energies of the two pucks.

$$K_{before} = \frac{1}{2}(5.0\ kg)(2.0\,m/s)^2 + \frac{1}{2}(5.0\ kg)(0.0\,m/s)^2$$

$$K_{before} = 10\ J$$

- Calculate the total kinetic energy of the system after the collision.

Again, let's sum the kinetic energies of the two pucks.

$$K_{after} = \frac{1}{2}(5.0\,kg)(\sqrt{3}\,m/s)^2 + \frac{1}{2}(5.0kg)(1.0\,m/s)^2$$

$$K_{after} = 10J$$

- Was the collision of the pucks elastic or inelastic? How do you know?

This was an elastic collision because the initial energy of the system is equal to the final energy of the system. No energy was lost due to non-conservative work.

- Suppose I aimed a fan at the pucks that exerted a constant, non-zero force on each of them along the -$\hat{y}$-direction. Could we still use the conservation of momentum to solve for the final velocities? Why or why not? Hint: The fan produces a non-zero, external force acting on the system.

When we use conservation of momentum, we assume there are no external forces acting on the system. Since we now have an external force, the law no longer applies.

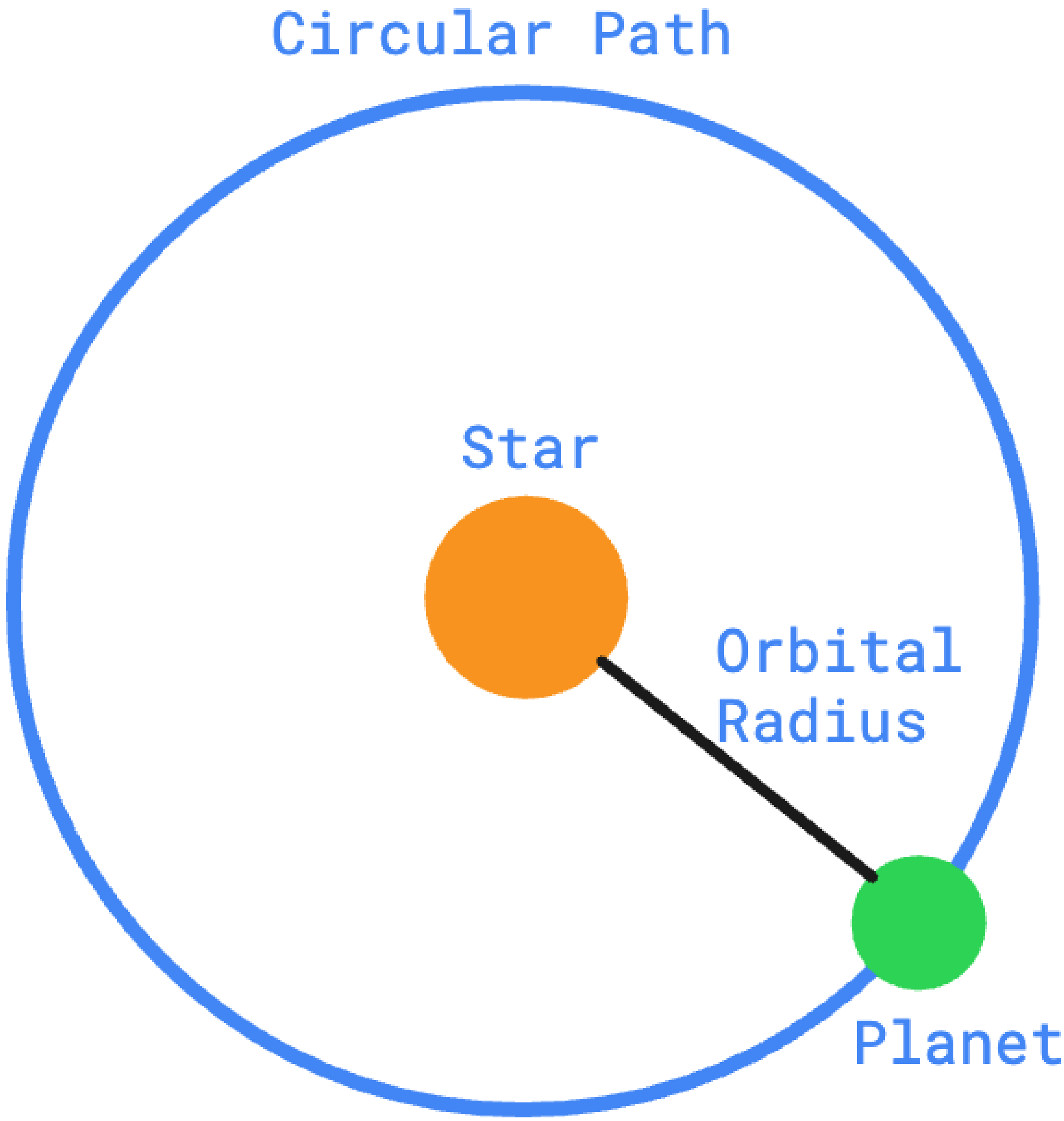
Problem 2

(45 points) A planet moves in a circular orbit about a star, with the star at the center of the circle. The mass of the planet, the mass of the star, and the radius of the orbit are all given below.

$M_{planet} = 6.0 \times 10^{23}\ kg$
$M_{star} = 2.0 \times 10^{32}\ kg$
$r_{orbit} = 2.5 \times 10^{10}\ m$

For this problem, I am going to let $M_{star} = M$, $M_{planet} = m$, and $r_{orbit} = R$.

- Sketch a diagram of the system. Label the planet, the star, the orbital radius, and the path that the planet takes around the star.



- What is the magnitude of the force of gravity that the star exerts on the planet?

We just care about the magnitude of the force of gravity in this problem. Thus, we can omit the unit vector and the negative sign.

$$F_G = \frac{GMm}{R^2}$$

$$F_G = \frac{\left(6.67 \times 10^{-11}\ N \cdot m^2/kg^2\right)\left(2.0 \times 10^{32}\ kg\right)\left(6.0 \times 10^{23}\ kg\right)}{\left(2.5 \times 10^{10}\ m\right)^2}$$

$$F_G = 1.28 \times 10^{25} N$$

- What is the magnitude of the force of gravity that the planet exerts on the star?

By Newton's laws, we know that the force is equal and opposite. Thus, the force of gravity is the same magnitude as the previous part.

- What is the speed of the planet if it is to maintain a stable circular orbit?

The gravitational force provides the centripetal force on the planet (we did this problem in class, in fact!).

$$F_G = \frac{mv^2}{R}$$

$$1.28 \times 10^{25} N = \frac{\left(6.0 \times 10^{23}\ kg\right) v^2}{2.5 \times 10^{10}\ m}$$

$$v = 730,479\ m/s$$

- What is the angular momentum of the planet in this orbit?

Let's start with the definition of angular momentum:

$$L = \vec{r} \times \vec{p} = R p_{plant} sin\theta$$

As per our diagram, the velocity of the planet is perpendicular to the radius of the orbit. Thus:

$$L = R p_{plant} = Rmv$$

$$L = \left(2.5 \times 10^{10}\ m\right)\left(6.0 \times 10^{23}\ kg\right)(730,479\ m/s)$$

$$L = 1.10 \times 10^{40}\ kg\, m^2/s$$

- What is the torque on the orbiting planet due to the force of gravity from the star? Use the position of the star as the origin for the torque.

Let's start with the definition of torque.

$$\tau = \vec{r} \times \vec{F} = R F_G sin\theta$$

The radius R points along the $+\hat{r}$ direction while the force of gravity points along the $-\hat{r}$ direction. Thus, the expression becomes:

$$\tau = RF_G\, sin\,(180^{\circ}) = 0\; N \cdot m$$

The torque turns out to be zero.

- What is the gravitational potential energy of the planet in this orbit? Write you answer in terms of M_{planet}, M_{star}, r_{orbit}, and constants.

There is actually no calculation needed for this part. Just write out the formula for the gravitational potential energy of a massive body. Don't forget the negative sign!

$$U_G = -\frac{GMm}{R}$$

- What is the kinetic energy of the planet in this orbit? Write you answer in terms of M_{planet}, M_{star}, r_{orbit}, and constants.

Recall from above that $F_G = mv^2/R$. We can re-write this as $v^2 = F_G R/m$. Then:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{F_G R}{m}\right) = \frac{GMm}{2R}$$

- What is the total mechanical energy of the orbiting planet? Write you answer in terms of M_{planet}, M_{star}, r_{orbit}, and constants.

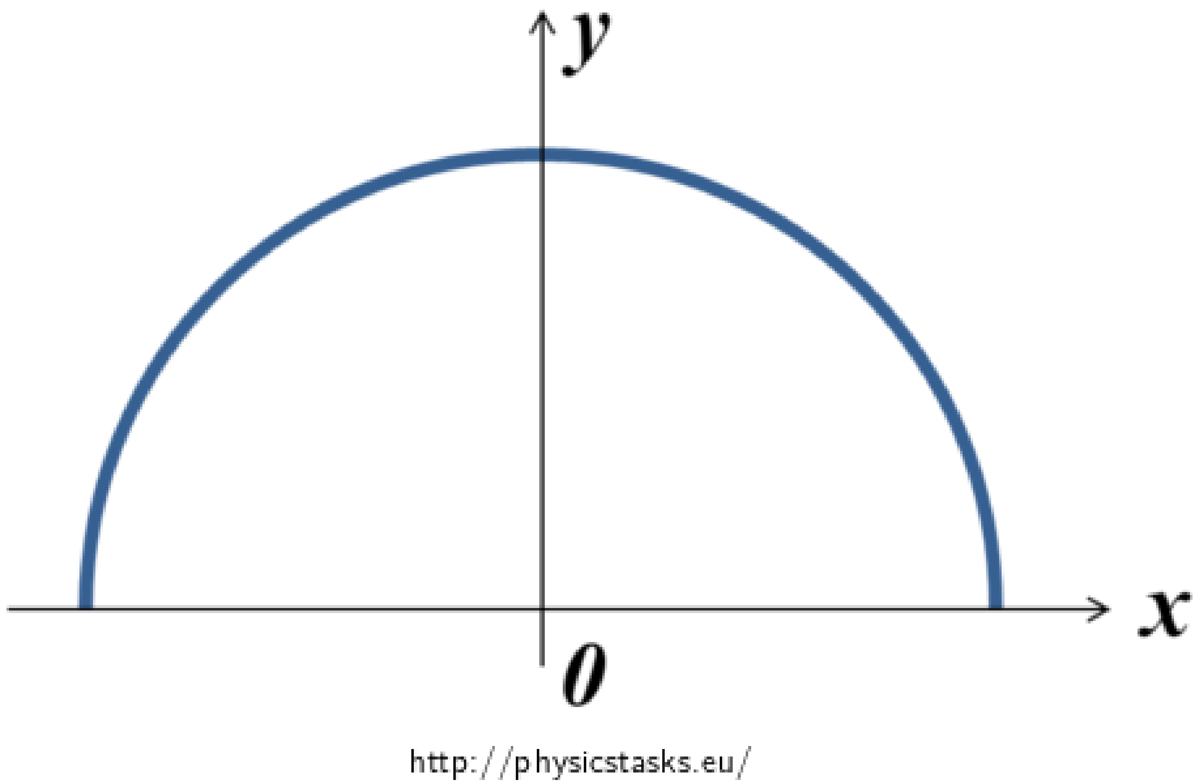
Mechanical energy is just potential energy plus kinetic energy.

$$U_{mechanical} = U_G + K = -\frac{GMm}{R} + \frac{GMm}{2R} = -\frac{GMm}{2R}$$

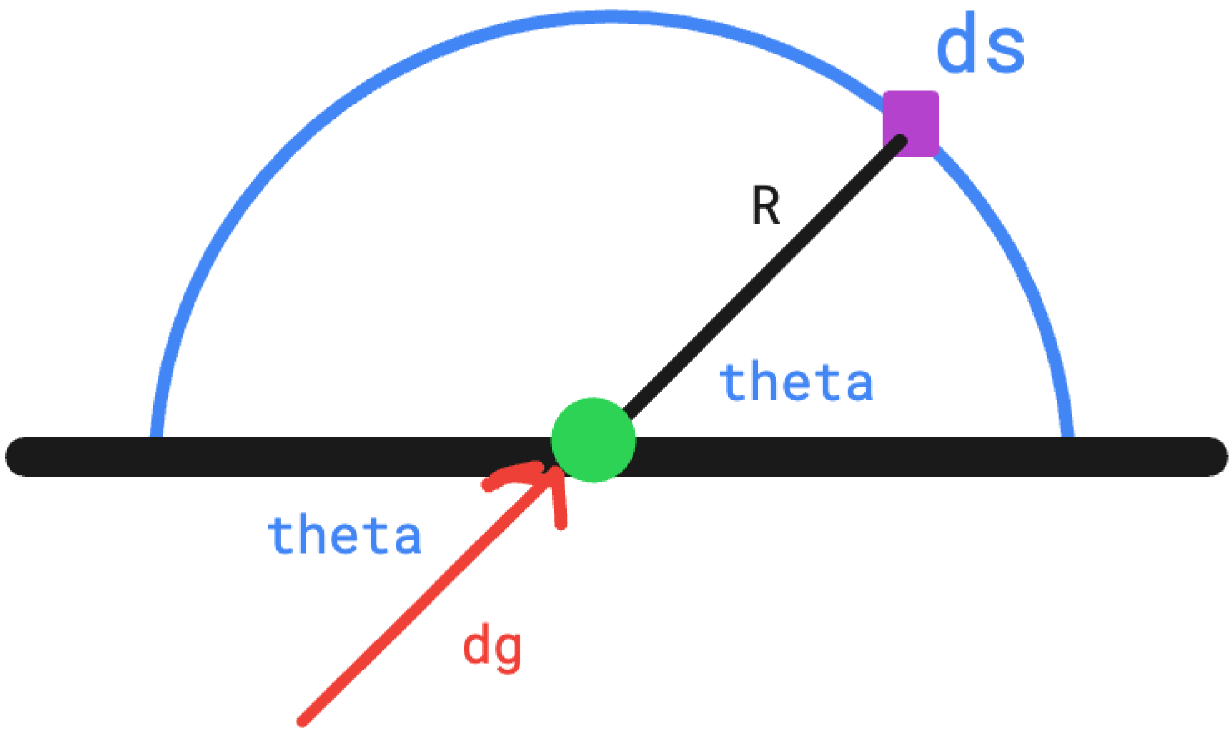
Problem 3

(15 points) A half-ring has a total mass of M and a radius of R, as shown in the diagram. Calculate the gravitational field at the center of the ring (i.e. - the origin of the xy-plane).

Hint: I would recommend putting everything in terms of constants and the angle θ and then integrating with respect to $d\theta$. Remember that arc-length (s) is related to angle (θ) by $s = R\theta$. This means that $ds = Rd\theta$.



Let's start by drawing a diagram. Remember, we cut the half-ring up into tiny pieces, each with a length ds. The gravitational field of a tiny piece of the ring dg is shown in the diagram.



The mass density (λ) of the ring is just the total mass divided by the total length. Thus, $\lambda = M/\pi R$.

From the mass density, we can determine the mass of a tiny piece of the ring, ds. It is given by $dm = \lambda ds$. We can express this in terms of $d\theta$ by using $ds = Rd\theta \Rightarrow dm = \lambda R d\theta$.

The distance from a tiny piece ds or $d\theta$ from the center of the circle is constant - it is always just R.

The angle θ in the diagram above can be used to break the field up into a horizontal and vertical component. Notice how the horizontal components from one side of the ring cancel out those from the other side of the ring. Thus, we only care about the gravitational field along the vertical direction. This corresponds to $sin\theta\hat{y}$.

Let's put everything together:

$$d\vec{g} = -\frac{Gdm}{r^2}\hat{r}$$

$$d\vec{g} = -\frac{G\lambda R d\theta}{R^2} sin\theta \left(\hat{y}\right)$$

$$d\vec{g} = -\frac{G\lambda d\theta}{R} sin\theta \left(\hat{y}\right)$$

Our limits of integration are from angle zero to π since this is a semi-circle.

$$\int_0^{\vec{g}} d\vec{g} = \int_0^{\pi} -\frac{G\lambda d\theta}{R} sin\theta \left(\hat{y}\right)$$

$$\vec{g} = -\frac{G\lambda}{R} \left(\hat{y}\right) \int_0^{\pi} sin\theta d\theta$$

$$\vec{g} = -\frac{G\lambda}{R} \left(\hat{y}\right) [-cos\theta]_0^{\pi}$$

$$\vec{g} = -\frac{G\lambda}{R} \left(\hat{y}\right) [-1 - 1]$$

$$\vec{g} = -\frac{-2G\lambda}{R} \left(\hat{y}\right)$$

$$\vec{g} = \frac{2G\lambda}{R} \left(\hat{y}\right)$$

We can substitute $\lambda = M/\pi R$ to get:

$$\vec{g} = \frac{2GM}{\pi R^2} \left(\hat{y}\right)$$